

Spaces of Incompressible Surfaces

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The homotopy type of the space of PL homeomorphisms of a Haken 3-manifold was computed in [H1], and with the subsequent proof of the Smale conjecture in [H2], the computation carried over to diffeomorphisms as well. These results were also obtained independently by Ivanov [I1,I2]. The main step in the calculation in [H1], though not explicitly stated in these terms, was to show that the space of embeddings of an incompressible surface in a Haken 3-manifold has components which are contractible, except in a few special situations where the components have a very simple noncontractible homotopy type. The purpose of the present note is to give precise statements of these embedding results along with simplified proofs using ideas from [H3]. We also rederive the calculation of the homotopy type of the diffeomorphism group of a Haken 3-manifold.

Let M be an orientable compact connected irreducible 3-manifold, and let S be an incompressible surface in M , by which we mean:

- S is a compact connected surface, not S^2 , embedded in M properly, i.e., $S \cap \partial M = \partial S$.
- The normal bundle of S in M is trivial. Since M is orientable, this just means S is orientable.
- The inclusion $S \hookrightarrow M$ induces an injective homomorphism on π_1 .

We denote by $E(S, M \text{ rel } \partial S)$ the space of smooth embeddings $S \rightarrow M$ agreeing with the given inclusion $S \hookrightarrow M$ on ∂S . The group $\text{Diff}(S \text{ rel } \partial S)$ of diffeomorphisms $S \rightarrow S$ restricting to the identity on ∂S acts freely on $E(S, M \text{ rel } \partial S)$ by composition, with orbit space the space $P(S, M \text{ rel } \partial S)$ of subsurfaces of M diffeomorphic to S by a diffeomorphism restricting to the identity on ∂S . We think of points of $P(S, M \text{ rel } \partial S)$ as “positions” or “placements” of S in M . There is a fibration

$$\text{Diff}(S \text{ rel } \partial S) \longrightarrow E(S, M \text{ rel } \partial S) \longrightarrow P(S, M \text{ rel } \partial S)$$

giving rise to a long exact sequence of homotopy groups. It is known that $\pi_i \text{Diff}(S \text{ rel } \partial S)$ is zero for $i > 0$, except when S is the torus T^2 and $i = 1$, when the inclusion $T^2 \hookrightarrow \text{Diff}(T^2)$ as rotations induces an isomorphism on π_1 . So the higher homotopy groups of $E(S, M \text{ rel } \partial S)$ and $P(S, M \text{ rel } \partial S)$ are virtually identical.

Theorem 1. *Let S be an incompressible surface in M . Then:*

- (a) $\pi_i P(S, M \text{ rel } \partial S) = 0$ for all $i > 0$ unless $\partial S = \emptyset$ and S is the fiber of a surface bundle structure on M . In this exceptional case the inclusion $S^1 \hookrightarrow P(S, M)$ as the fiber surfaces induces an isomorphism on π_i for all $i > 0$.
- (b) $\pi_i E(S, M \text{ rel } \partial S) = 0$ for all $i > 0$ unless $\partial S = \emptyset$ and S is either a torus or the fiber of a surface bundle structure on M . In these exceptional cases $\pi_i E(S, M) = 0$ for all $i > 1$. In the surface bundle case the inclusion of the subspace consisting of embeddings with image a fiber induces an isomorphism on π_1 . When S is a torus but not the fiber of a surface bundle structure, the inclusion of the subspace consisting of embeddings with image equal to the given S induces an isomorphism on π_1 .

Here it is understood that $\pi_i E(S, M \text{ rel } \partial S)$ and $\pi_i P(S, M \text{ rel } \partial S)$ are to be computed at the basepoint which is the given inclusion $S \hookrightarrow M$.

It is not hard to describe precisely what happens in the exceptional cases of (b). In the surface bundle case consider the exact sequence

$$0 \longrightarrow \pi_1 \text{Diff}(S) \longrightarrow \pi_1 E(S, M) \longrightarrow \pi_1 P(S, M) \xrightarrow{\partial} \pi_0 \text{Diff}(S)$$

By part (a) we have $\pi_1 P(S, M) \approx \mathbb{Z}$. The boundary map takes a generator of this \mathbb{Z} to the monodromy diffeomorphism defining the surface bundle. If this monodromy has infinite order in $\pi_0 \text{Diff}(S)$ then the boundary map is injective so $\pi_1 E(S, M) \approx \pi_1 \text{Diff}(S)$. If the monodromy has finite order in $\pi_0 \text{Diff}(S)$, it is isotopic to a periodic diffeomorphism of this order, as Nielsen showed. Hence M is Seifert-fibered with coherently oriented fibers, and there is an action of S^1 on M rotating circle fibers, and taking fibers of the surface bundle structure to fibers. The orbit of the given embedding of S under this action then generates a \mathbb{Z} subgroup of $\pi_1 E(S, M)$, which is all of $\pi_1 E(S, M)$ unless S is a torus, in which case it is easy to see that $\pi_1 E(S, M) \approx \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

Using these results we can deduce:

Theorem 2. *If M is an orientable Haken manifold then $\pi_i \text{Diff}(M \text{ rel } \partial M) = 0$ for all $i > 0$ unless M is a closed Seifert manifold with coherently orientable fibers. In the latter case the inclusion $S^1 \hookrightarrow \text{Diff}(M)$ as rotations of the fibers induces an isomorphism on π_i for all $i > 0$, except when M is the 3-torus, in which case the circle of rotations $S^1 \hookrightarrow \text{Diff}(M)$ is replaced by the 3-torus of rotations $M \hookrightarrow \text{Diff}(M)$.*

Proof: Suppose first that $\partial M \neq \emptyset$, hence M is automatically Haken if it is irreducible. By the theory of Haken manifolds, there exists an incompressible surface $S \subset M$ with $\partial S \neq \emptyset$, in fact with ∂S representing a nonzero class in $H_1(\partial M)$. Consider the fibration

$$\text{Diff}(M \text{ rel } \partial M \cup S) \longrightarrow \text{Diff}(M \text{ rel } \partial M) \longrightarrow E(S, M \text{ rel } \partial S)$$

The fiber can be identified with $Diff(M' \text{ rel } \partial M')$ where M' is the compact manifold, possibly disconnected, obtained by splitting M along S . If we know the theorem holds for M' , then from the long exact sequence of homotopy groups of the fibration, together with part (b) of the preceding theorem, we deduce that the theorem holds for M . Again by Haken manifold theory, there is a finite sequence of such splitting operations reducing M to a disjoint union of balls. By the Smale conjecture the theorem holds for balls, so by induction the theorem holds for M .

When M is a closed Haken manifold we again consider the fibration displayed above, with S now a closed incompressible surface. If we are not in the exceptional cases that $\pi_1 E(S, M)$ is nonzero, described in the paragraph before Theorem 2, the arguments in the preceding paragraph apply since M' is a nonclosed Haken manifold, for which the theorem has already been proved.

There remain the cases that $\pi_1 E(S, M) \neq 0$.

(i) If S is the fiber of a surface bundle structure on M but not a torus, the result follows from the remarks preceding Theorem 2, describing how a generator of $\pi_1 E(S, M) \approx \mathbb{Z}$ is represented by the orbit of the inclusion $S \hookrightarrow M$ under the S^1 action rotating fibers of the Seifert fibering.

(ii) If S is a torus but not the fiber of a surface bundle, we have $\pi_1 E(S, M) \approx \mathbb{Z} \times \mathbb{Z}$, the rotations of S . Under the boundary map $\pi_1 E(S, M) \rightarrow \pi_0 Diff(M \text{ rel } S)$ each loop of rotations goes to a Dehn twist on one side of S and its inverse twist on the other side. The behavior of such twists in the mapping class group of the manifold M' is well known; see e.g. [HM]. In particular, the boundary map is injective unless M is orientably Seifert-fibered with S a union of fibers, in which case the kernel of the boundary map is \mathbb{Z} represented by rotations of the circle fibers.

(iii) If S is the torus fiber of a surface bundle we have to consider the phenomena in both (i) and (ii). The monodromy diffeomorphism of the torus fiber defining the bundle is an element of $SL_2(\mathbb{Z})$. Note that a loop of rotations of S as in (ii) lies in the kernel of the boundary map iff the rotations are in the direction of a curve in S fixed by the monodromy. If the monodromy is trivial, M is the 3-torus and we have contributions to $\pi_1 Diff(M)$ from both (i) and (ii), so we get $M \subset Diff(M)$ inducing an isomorphism on π_i for $i > 0$. If the monodromy is nontrivial and of finite order, it has no real eigenvalues, so the boundary map in (ii) is injective and we get only the $S^1 \subset Diff(M)$ as in (i). If the monodromy is of infinite order and has 1 as an eigenvalue, it is a Dehn twist of the torus fiber, so we get $\pi_1 Diff(M) \approx \mathbb{Z}$ and this loop of diffeomorphisms is realized by rotating fibers of a fibering of M by circles in the eigendirection in the torus fibers of M . \square

Proof of Theorem 1.

We will first prove statement (b) when $\partial S \neq \emptyset$, which suffices to deduce Theorem 2 in the nonclosed case. Then we will prove (a), and finally the remaining cases of (b).

Let $f_t: S \rightarrow M$, $t \in D^i$, be a family of embeddings representing an element of $\pi_i E(S, M \text{ rel } \partial S)$. We assume $i > 0$, so all the surfaces $f_t(S)$ are isotopic rel ∂S to the given inclusion $S \hookrightarrow M$, hence are incompressible also. Let $S \times I$ be a collar on one side of $S = S \times \{0\}$ in M . By Sard's theorem, $f_t(S)$ is transverse to $S \times \{x\}$ for almost all $x \in I$. Transversality is preserved under small perturbations, so, since D^i is compact, we can choose a finite cover of D^i by open sets U_j such that $f_t(S)$ is transverse to a slice $S_j = S \times \{x_j\} \subset S \times I$ for all $t \in U_j$. Then $f_t(S) \cap S_j$ is a finite collection of disjoint circles which vary by isotopy as t ranges over U_j . The main work will be to deform the family f_t to eliminate these circles, for all t and j simultaneously (after choosing the slices S_j and the open sets U_j a little more carefully).

Step 1. The aim here is to eliminate all the nullhomotopic circles. Let \mathcal{C}_t^j be the collection of circles of $f_t(S) \cap S_j$ which are homotopically trivial in M and hence bound disks in both $f_t(S)$ and S_j , by incompressibility. Let $\mathcal{C}_t = \bigcup_j \mathcal{C}_t^j$, the union over those j 's such that $t \in U_j$. We would like to construct a family of functions φ_t assigning a value $\varphi_t(C) \in (0, 1)$ to each circle $C \in \mathcal{C}_t$ such that:

- (1) $\varphi_t(C)$ varies continuously with $t \in U_j$ for each $C \in \mathcal{C}_t^j$.
- (2) $\varphi_t(C) < \varphi_t(C')$ whenever the disk in $f_t(S)$ bounded by C is contained in the disk bounded by C' .
- (3) φ_t is injective for each t , so $\varphi_t(C) \neq \varphi_t(C')$ if C and C' are distinct circles of \mathcal{C}_t .

Achieving (1) and (2) is not hard. For example, one can take $\varphi_t(C)$ to be the area of the disk in $f_t(S)$ bounded by C , with respect to some metric on M . To achieve (3) takes more work. First replace each S_j by $2i + 1$ nearby slices $S_{jk} = S \times \{x_{jk}\}$, so that each circle of $f_t(S) \cap S_j$ is replaced by $2i + 1$ nearby circles of $f_t(S) \cap S_{jk}$. Define φ_t on each of these new circles of $f_t(S) \cap S_{jk}$ to have value near the value of the original φ_t on the nearby circle of $f_t \cap S_j$, so that (1) and (2) are still satisfied for all the new circles. Perturb the new functions $t \mapsto \varphi_t(C)$ so that the solution set of each equation $\varphi_t(C) = \varphi_t(C')$ is a codimension-one submanifold of D^i , and so that these codimension-one submanifolds have general position intersections with each other, and in particular so that at most i such equations are satisfied for each t . Then for each t the perturbed φ_t is injective on the complement of a set of at most $2i$ circles. This means that if we delete from \mathcal{C}_t those circles on which φ_t is not injective, there exists for each t a slice S_{jk} from which no circles have been deleted. This slice has the same property for nearby t . Let U_{jk} be the subspace

of U_j consisting of points t for which no circles of \mathcal{C}_t in S_{jk} are deleted. Then the open cover $\{U_{jk}\}$ of D^i and the associated slices S_{jk} satisfy (1)-(3). We relabel these as U_j and S_j .

We would like to deform the family f_t , $t \in D^i$, so as to eliminate all the circles of \mathcal{C}_t without introducing new circles of $f_t(S) \cap S_j$ for $t \in U_j$. Consider first a fixed value of t and a circle $C \in \mathcal{C}_t$ for which $\varphi_t(C)$ is minimal. Thus C bounds a disk $D \subset f_t(S)$ disjoint from all other circles of \mathcal{C}_t . In particular, $D \cap S_j = C$, where $C \subset S_j$. By incompressibility of S_j , C also bounds a disk $D_j \subset S_j$, and the sphere $D \cup D_j$ bounds a ball $B \subset M$ since M is irreducible. We can isotope f_t to eliminate C from $f_t(S) \cap S_j$ by isotoping D across B to D_j , and slightly beyond. This isotopy extends to an ambient isotopy of M supported near B which also eliminates any circles of $f_t(S) \cap S_j$ which happen to lie inside D_j ; we call such circles *secondary*, in contrast to C itself which we call *primary*. Since D and D_j are disjoint from S_k if $k \neq j$ and $t \in U_k$, so is B , so this isotopy leaves circles of $f_t(S) \cap S_k$ unchanged if $k \neq j$ and $t \in U_k$. Hence we can iterate the process, eliminating in turn each remaining circle of \mathcal{C}_t with smallest φ_t value. Thus we construct an isotopy f_{tu} , $0 \leq u \leq 1$, of $f_t = f_{t0}$ eliminating each primary circle $C \in \mathcal{C}_t$ during the u -interval $[\varphi_t(C), \varphi_t(C) + \varepsilon]$, for some fixed ε , along with any secondary circles associated to C .

This isotopy f_{tu} will not depend continuously on t since as t moves from U_j to the complement of U_j , we suddenly stop performing the isotopies eliminating the circles of $f_t(S) \cap S_j$. This problem is easy to correct by the following truncation process. For each U_j choose a map $\psi_j : D^i \rightarrow [0, 1]$ which is 0 outside U_j and 1 inside a slightly smaller open set U'_j in U_j , such that the U'_j 's still cover D^i . Then modify the construction of f_{tu} by performing the isotopies eliminating primary circles of $f_t(S) \cap S_j$ only for $u \leq \psi_j(t)$. As observed earlier, the isotopy eliminating a primary circle of $f_t(S) \cap S_j$ does not affect circles of $f_t(S) \cap S_k$ for $k \neq j$ with $t \in U_k$, so truncating such an isotopy at $u = \psi_j(t)$ creates no problems for continuing the construction of f_{tu} for $u > \psi_j(t)$ to eliminate circles of $f_t(S) \cap S_k$.

There is one other reason why f_{tu} , as described so far, may not depend continuously on t , namely, there is a choice in the isotopy eliminating a primary circle, and these choices need to be made continuously in t . We may specify an isotopy eliminating a circle C by the following process. First enlarge the disk D slightly to a disk $D' \subset f_t(S)$, with $\partial D'$ contained in a nearby parallel copy S'_j of S_j , then choose a collar on D' containing B , i.e., an embedding $\chi : D' \times I \rightarrow M$ with $\chi|_{D' \times \{0\}}$ the identity and $B \subset \chi(D' \times [0, 1/2]) \subset \chi(D' \times I) \subset B'$ where B' is a ball constructed like B using S'_j in place of S_j . We may also assume $\chi(\partial D' \times I) \subset S'_j$. Then an isotopy eliminating C is obtained by pushing

points $x \in D'$ along the lines $\chi(\{x\} \times I)$, with this motion damped down near $\partial D'$.

The space of such collars is contractible. Namely, use one collar η to produce an isotopy $h_s: B' \rightarrow B'$ moving $\eta(D' \times \{0\})$ to $\eta(D' \times \{1\})$ in the obvious way. Then for an arbitrary collar χ , construct a deformation χ_s of χ consisting of $h_s\chi$ together with a portion of η . (To make χ_s smooth at the junction of these two pieces one can require all χ 's to have the same derivative along $D' \times \{0\}$.) Then χ_1 contains the collar η , so we can deform χ_1 to η by gradually truncating the part outside η .

To make the dependence of f_{tu} on u explicit we may proceed as follows. We may assume the truncation functions $\psi_j(t)$ are piecewise linear, and we may perturb the functions $\varphi_t(C)$ to be piecewise linear functions of t also. Then we may choose a triangulation of D^i so that each simplex is contained in some U_j and so that the solution set of each equation $\psi_j(t) = \varphi_t(C)$ is a subcomplex of the triangulation. Now we construct the isotopies f_{tu} inductively over skeleta of the triangulation. Assume that f_{tu} has already been constructed over the boundary of a k -simplex Δ^k , and assume that over Δ^k itself we have already constructed f_{tu} for $t \leq \varphi_t(C)$, for some primary circle $C \in \mathcal{C}_t^j$. If $\psi_j(t) \leq \varphi_t(C)$ over Δ^k , the induction step is vacuous, so we may suppose $\psi_j(t) \geq \varphi_t(C)$ for $t \in \Delta^k$. By induction, for $t \in \partial\Delta^k$ we have already chosen collars χ_t for the disk D' , as above, depending continuously on t , and since the space of collars is contractible we can extend these collars over Δ^k . This allows the induction step to be completed, eliminating C over Δ^k , with the elimination isotopy truncated if ψ_t so dictates.

Step 2. We show how to finish the proof of (b) if S is a disk. After Step 1, all the circles of $f_t(S) \cap S_j$ have been eliminated for all $t \in U_j$ and all j . By averaging the slices $S_j = S \times \{x_j\}$ via a partition of unity subordinate to the U_j 's we can choose a continuously varying slice $S_t = S \times \{x_t\}$ disjoint from $f_t(S)$; here we use the fact that if $f_t(S)$ is disjoint from $S \times \{x\}$ and from $S \times \{y\}$ then it is disjoint from $S \times [x, y]$ since $f_t(S)$ is connected and its boundary, which is non-empty, lies in $S = S \times \{0\}$. Having $f_t(S)$ disjoint from S_t for all t , we can then by isotopy extension isotope the family f_t so that $f_t(S)$ is disjoint from $S \times \{1\}$ for all t .

The proof can now be completed as follows. The space E of embeddings $S \times I \rightarrow M$ agreeing with the given embedding on $S \times \{1\} \cup \partial S \times I$ fits into a fibration:

$$Diff(S \times I \text{ rel } \partial) \longrightarrow E \longrightarrow E(S, M - S \times \{1\} \text{ rel } \partial S)$$

It is elementary that E has trivial homotopy groups since one can canonically isotope an embedding in E so that it equals the given embedding on a neighborhood of $S \times \{1\} \cup \partial S \times I$, then gradually excise from the embedding everything but this neighborhood. The fiber

$Diff(S \times I \text{ rel } \partial)$ also has trivial homotopy groups by the Smale conjecture since S is a disk. Thus $\pi_i E(S, M - S \times \{1\} \text{ rel } \partial S)$ vanishes for $i > 0$, and the first part of the proof shows that this group maps onto $\pi_i E(S, M \text{ rel } \partial S)$, so the latter group also vanishes. When S is a disk this argument also applies for $i = 0$ since $\pi_0 E(S, M - S \times \{1\} \text{ rel } \partial S) = 0$ by the irreducibility of M .

Step 3 is to eliminate the remaining circles of $f_t(S) \cap S_j$ for all $t \in U_j$ and all j , in the case $\partial S \neq \emptyset$. The role of the ball B in Step 1 will be played by what we may call a *pinched product*. This is obtained from a product $W \times I$, where W is a nonclosed compact orientable surface, by collapsing each segment $\{w\} \times I$, $w \in \partial W$, to a point. Thus a pinched product P is a handlebody with its boundary decomposed as the union of two copies $\partial_+ P$ and $\partial_- P$ of the surface W , with $\partial_+ P$ and $\partial_- P$ intersecting only in their common boundary, a “corner” of ∂P .

If we can find a pinched product $P \subset M$ such that $\partial_+ P \subset f_t(S)$ and $\partial_- P \subset S_j$, then by isotoping f_t by pushing $\partial_+ P$ across P to $\partial_- P$, and slightly beyond, we eliminate the circles of $\partial_+ P \cap \partial_- P$ from $f_t(S) \cap S_j$, as well as any other circles of $f_t(S) \cap S_j$ which happen to lie in $\partial_- P$.

In order to locate such pinched products it is convenient to consider the covering space $p: (\widetilde{M}, \widetilde{x}_0) \rightarrow (M, x_0)$ corresponding to the subgroup $\pi_1(S, x_0)$ of $\pi_1(M, x_0)$, with respect to a basepoint $x_0 \in S$. There is then a homeomorphic copy \widetilde{S} of S in \widetilde{M} containing \widetilde{x}_0 . We can associate to \widetilde{M} a graph T having a vertex for each component of $\widetilde{M} - p^{-1}(S)$ and an edge for each component of $p^{-1}(S)$. This graph T is in fact a tree. For suppose γ is a loop in T based at a point in the edge corresponding to \widetilde{S} . This can be lifted to a loop $\widetilde{\gamma}$ in \widetilde{M} based at \widetilde{x}_0 . By the definition of \widetilde{M} , the loop $\widetilde{\gamma}$ is homotopic to a loop in \widetilde{S} , and this homotopy projects to a homotopy from γ to a trivial loop.

In the same way, the surface S_j parallel to S determines a tree T_j canonically isomorphic to T . The lift \widetilde{S} lies in a component of $\widetilde{M} - p^{-1}(S_j)$ corresponding to a base vertex of T_j , and hence to a base vertex of T which is independent of j .

The family $f_t(S)$ is an i -parameter isotopy of S , so there are lifts $\widetilde{f}_t: S \rightarrow \widetilde{M}$ such that $\widetilde{f}_t(S)$ forms an i -parameter isotopy of \widetilde{S} . The lifts $\widetilde{f}_t(S)$ and the parameter domain being compact, we can choose a vertex v of T farthest from the base vertex among all vertices corresponding to components of $\widetilde{M} - p^{-1}(S_j)$ which meet $\widetilde{f}_t(S)$ as t varies over S^k and j varies arbitrarily. Let \widetilde{V}_j be the closure of the component of $\widetilde{M} - p^{-1}(S_j)$ corresponding to v , with \widetilde{S}_j its boundary component in the direction of \widetilde{S} . Let $\widetilde{\mathcal{C}}_t^j$ be the collection of components of $\widetilde{f}_t(S) \cap \widetilde{V}_j$ and let \mathcal{C}_t^j be the (diffeomorphic) images of these components in $f_t(S)$. Let $\mathcal{C}_t = \bigcup_j \mathcal{C}_t^j$, the union over j such that $t \in U_j$.

Lemma. For each surface $C \in \mathcal{C}_t^j$ there is a pinched product $P \subset M$ with $\partial_+ P = C$ and $\partial_- P \subset S_j$.

Proof: This is proved in II.5 of [L], but let us sketch an argument which may be more direct. First observe that $\pi_1(\tilde{V}_j, \tilde{S}_j) = 0$, otherwise $\pi_1(\tilde{S}_j) \rightarrow \pi_1(\tilde{V}_j)$ would not be surjective and hence $\pi_1(\tilde{S}) \rightarrow \pi_1(\tilde{M})$ could not be an isomorphism. (This uses the fact that the components of $p^{-1}(S_j)$ are incompressible in \tilde{M} .) Next, let V_j be M split along S_j , so we have a covering space $p: (\tilde{V}_j, \tilde{S}_j) \rightarrow (V_j, S_j)$. Since the inclusion $(C, \partial C) \hookrightarrow (V_j, S_j)$ lifts to $(\tilde{V}_j, \tilde{S}_j)$, the fact that $\pi_1(\tilde{V}_j, \tilde{S}_j) = 0$ implies that $\pi_1(C, \partial C) \rightarrow \pi_1(V_j, S_j)$ is zero. Thus there is a map $F: D^2 \rightarrow V_j$ giving a homotopy from a path α in C representing a nontrivial element of $\pi_1(C, \partial C)$ to a path β in S_j . We would like to improve F to be an embedding with $F(D^2) \cap C = \alpha$. To do this, first perturb F to be transverse to C . We can modify F to eliminate any circles of $F^{-1}(C)$, using the fact that C is incompressible in V_j . Arcs of $F^{-1}(C)$, other than α , which are trivial in $\pi_1(C, \partial C)$ can be eliminated similarly. An outermost remaining arc of $F^{-1}(C)$ can be used as a new α for which $F(D^2) \cap C = \alpha$. Now we apply the loop theorem to replace F by an embedding with the same properties. Having this embedded disk, we can isotope C by pushing α across the disk, surgering C to a simpler surface C' which, by induction on the complexity of C , splits off a pinched product P' from V_j ; the induction starts with the case that C is a disk, where incompressibility of S_j and irreducibility of M gives the result. We recover C from C' by adjoining a “tunnel.” If this tunnel lies outside P' , then the tunnel enlarges P' to a pinched product P split off from V_j by C , as desired. The other alternative, that the tunnel lies inside P' , cannot occur since C would then be compressible. \square

Continuing with the main line of the proof, we would like to choose functions $\varphi_t: \mathcal{C}_t \rightarrow (0, 1)$ satisfying:

- (1) $\varphi_t(C)$ varies continuously with $t \in U_j$ for each $C \in \mathcal{C}_t^j$.
- (2) $\varphi_t(C) < \varphi_t(C')$ if $C \subset C'$.
- (3) φ_t is injective for each t .

As before, (1) and (2) are easy to arrange, and then (3) is achieved by replacing each S_j with a number of nearby copies of itself and rechoosing the cover $\{U_j\}$.

Having functions φ_t satisfying (1)-(3), we follow the same scheme as in Step 1 to construct an isotopy f_{tu} of f_t eliminating components $C \in \mathcal{C}_t$ during the corresponding u -intervals $[\varphi_t(C), \varphi_t(C) + \varepsilon]$. Again there are primary and secondary components, and only the primary components need be dealt with explicitly.

The end result of this family of isotopies f_{tu} is that the vertex v is no longer among the vertices of T corresponding to components of $\tilde{M} - p^{-1}(S_j)$ meeting $\tilde{f}_t(S)$ for $t \in U_j$,

and no new such vertices have been introduced. So by iteration of the process we eventually reach the situation that $f_t(S)$ is disjoint from S_j for all $t \in U_j$ and all j .

Step 4. We can now finish the proof of (b) in the case that $\partial S \neq \emptyset$ by the argument in Step 2. The assumption that S was a disk rather than an arbitrary compact orientable surface with non-empty boundary was used in Step 2 only to deduce that $Diff(S \times I \text{ rel } \partial)$ had trivial homotopy groups, but the case $S = D^2$ suffices to show this in the present case since the handlebody $S \times I$ can be reduced to a ball by cutting along a collection of disjoint disks; see the proof of Theorem 2.

Step 5. Now we prove (a) when $\partial S \neq \emptyset$. Steps 1 and 3 work equally well in this case, with images of embeddings instead of actual embeddings. The only modification needed in steps 2 and 4 is to show that $P(S, M - S \times \{1\} \text{ rel } \partial S)$ has trivial π_i for $i > 0$. The basepoint component of $P(S, M - S \times \{1\} \text{ rel } \partial S)$ can be identified with the base space in the following fibration, where $A = S \times \{1\} \cup \partial S \times I$.

$$Diff(S \times I \text{ rel } A) \longrightarrow E(S \times I, M \text{ rel } A) \longrightarrow E(S \times I, M \text{ rel } A)/Diff(S \times I \text{ rel } A)$$

The total space $E(S \times I, M \text{ rel } A)$ is contractible, as in Step 2. And the fiber has trivial homotopy groups, as one can see from the fibration

$$Diff(S \times I \text{ rel } \partial(S \times I)) \longrightarrow Diff(S \times I \text{ rel } A) \longrightarrow Diff(S \times \{0\} \text{ rel } \partial S \times \{0\})$$

Step 6. This is to prove (a) when $\partial S = \emptyset$. Let Σ_t be a family of surfaces representing an element of $\pi_i P(S, M)$. Steps 1 and 3 work when $\partial S = \emptyset$, so we may assume Σ_t is disjoint from S_j for all $t \in U_j$ and all j . However, this does not imply Σ_t is disjoint from the regions between S_j 's as it did when $\partial S \neq \emptyset$. Instead, let us look at the lifts $\tilde{\Sigma}_t$ to \tilde{M} , which are well defined at least when the parameter domain S^i is simply-connected, i.e., when $i > 1$. Let \tilde{V}_j and \tilde{S}_j be defined as in Step 3. Then $\tilde{\Sigma}_t \subset \tilde{V}_j$ for $t \in U_j$. If \tilde{S} is not contained in \tilde{V}_j , then \tilde{S} and $\tilde{\Sigma}_t$ are disjoint for $t \in U_j$, hence, since they are isotopic, the region between them in \tilde{M} is a product $\tilde{S} \times I$. The surface \tilde{S}_j lies in this region, hence must be compact too. Since \tilde{S}_j is an incompressible surface in the interior of the product $\tilde{S} \times I$, the region between \tilde{S}_j and $\tilde{\Sigma}_t$ must be a product, projecting diffeomorphically to a product region between S_j and Σ_t . We can use such products in place of the pinched products in Step 3, and thus isotope the family Σ_t to a new family for which \tilde{V}_j does contain \tilde{S} . Then we can easily isotope the family Σ_t to be disjoint from $S \times \{1\}$ for all t and apply the argument in Step 5 to finish the proof that $\pi_i P(S, M) = 0$ for $i > 1$.

When $i = 1$ we can replace the parameter domain S^1 by I and then the lifts $\tilde{\Sigma}_t$ exist, with $\tilde{\Sigma}_0 = \tilde{S}$. The difficulty is that $\tilde{\Sigma}_1$ may be a different lift of S from \tilde{S} . If this

happens, then in the notation of the preceding paragraph, \tilde{V}_j will not contain \tilde{S} . In this case the product $\tilde{S} \times I$ will contain another lift of S . The region between \tilde{S} and such a lift will also be a product $\tilde{S} \times I$, which means that M must be a surface bundle with S as fiber. Then we have a surjective homomorphism $\pi_1 P(S, M) \rightarrow \mathbb{Z}$ which measures which lift of S to $\tilde{M} = S \times \mathbb{R}$ the surface $\tilde{\Sigma}_1$ is. On the kernel of this homomorphism the arguments of the preceding paragraph apply, and we deduce that this kernel is zero. Thus $\pi_1 P(S, M)$ is \mathbb{Z} , represented by fibers of the surface bundle. When M is not a surface bundle with fiber S we must have $\tilde{\Sigma}_1 = \tilde{S}$, and $\pi_1 P(S, M) = 0$.

Step 7. When $\partial S = \emptyset$ we can deduce (b) from (a) by looking at the long exact sequence of homotopy groups for the fibration $Diff(S) \rightarrow E(S, M) \rightarrow P(S, M)$. This is immediate except when M is a surface bundle with fiber S . In this special case, exactness at $\pi_1 E(S, M)$ implies that this fundamental group is represented by loops of embeddings $S \hookrightarrow M$ with image a fiber. \square

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